Chapter 11

Linear Equations

The most important task in technical computing.

I am thinking of two numbers. Their average is 3. What are the numbers? Please remember the first thing that pops into your head. I will get back to this problem in a few pages.

Solving systems of simultaneous linear equations is the most important task in technical computing. It is not only important in its own right, it is also a fundamental, and often hidden, component of other more complicated computational tasks.

The very simplest linear equations involve only one unknown. Solve

$$7x = 21$$

The answer, of course, is

$$x = \frac{21}{7} = 3$$

Now solve

$$ax = b$$

The answer, of course, is

$$x = \frac{b}{a}$$

But what if $a = 0$? Then we have to look at b. If $b \neq 0$ then there is no value of x that satisfies

$$0x = b$$

The solution does not exist. On the other hand, if $b = 0$ then any value of x satisfies

$$0x = 0$$

The solution is not unique. Mathematicians have been thinking about existence and uniqueness for centuries. We will see that these concepts are also important in modern technical computing.

Here is a toy story problem.

> Alice buys three apples, a dozen bananas, and one cantaloupe for $2.36. Bob buys a dozen apples and two cantaloupes for $5.26. Carol buys two bananas and three cantaloupes for $2.77. How much do single pieces of each fruit cost?

Let x_1, x_2, and x_3 denote the unknown price of each fruit. We have three equations in three unknowns.

$$\begin{aligned} 3x_1 + 12x_2 + \ x_3 &= 2.36 \\ 12x_1 \qquad\quad + 2x_3 &= 5.26 \\ 2x_2 + 3x_3 &= 2.77 \end{aligned}$$

Because matrix-vector multiplication has been defined the way it has, these equations can be written

$$\begin{pmatrix} 3 & 12 & 1 \\ 12 & 0 & 2 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2.36 \\ 5.27 \\ 2.77 \end{pmatrix}$$

Or, simply

$$Ax = b$$

where A is a given 3-by-3 matrix, b is a given 3-by-1 column vector, and x is a 3-by-1 column vector with unknown elements.

We want to solve this equation. If you know anything about matrices, you know that the equation can be solved using A^{-1}, the *inverse* of A,

$$x = A^{-1}b$$

This is a fine concept theoretically, but not so good computationally We don't really need A^{-1}, we just want to find x. If you do not know anything about matrices, you might be tempted to divide both sides of the equation by A.

$$x = \frac{b}{A}$$

This is a terrible idea theoretically – you can't divide by matrices – but it is the beginning of a good idea computationally.

To find the solution to a linear system of equations with MATLAB, start by entering the matrix of coefficients.

```
A = [3 12 1; 12 0 2; 0 2 3]
```

Since all the elements of `A` are integers, the matrix is printed with an integer format.

```
A =
     3    12     1
    12     0     2
     0     2     3
```

Next, enter the right hand side as a column vector.

```
b = [2.36 5.26 2.77]'
```

The elements of `b` are not integers, so the default format shows four digits after the decimal point.

```
b =
    2.3600
    5.2600
    2.7700
```

MATLAB has an output format intended for financial calculations, like this fruit price calculation. The command

```
format bank
```

changes the output to show only two digits after the decimal point.

```
b =
    2.36
    5.26
    2.77
```

In MATLAB the solution to the linear system of equations

$$Ax = b$$

is found using the *backslash* operator.

```
x = A\b
```

Think of this as "dividing" both sides of the equation by `A`. The result is

```
x =
    0.29
    0.05
    0.89
```

This give us the solution to our story problem – apples cost 29 cents each, bananas are a nickel each, and cantaloupes are 89 cents each.

Very rarely, systems of linear equations come in the form

$$xA = b$$

where b and x are row vectors. In this case, the solution is found using the forward slash operator.

```
x = b/A
```

The two operations `A\b` and `b/A` are sometimes called *left* and *right* matrix division. In both cases, the coefficient matrix is in the "denominator". For scalars, left and right division are the same thing. The quantities `7\21` and `21/7` are both equal to `3`.

Let's change our story problem a bit. Assume now that Carol buys six apples and one cantaloupe for $2.77. The coefficient matrix and right hand side become

```
A =
     3    12     1
    12     0     2
     6     0     1
```

and

```
b =
    2.36
    5.26
    2.77
```

At first glance, this does not look like much of a change. However,

```
x = A\b
```

produces

```
Warning: Matrix is singular to working precision.
x =
  NaN
 -Inf
  Inf
```

`Inf` and `-Inf` stand for plus and minus infinity and result from division of nonzero numbers by zero. `NaN` stands for "Not-a-Number" and results from doing arithmetic involving infinities.

The source of the difficulty is that the new information about Carol's purchase is inconsistent with the earlier information about Alice's and Bob's purchases. We have said that Carol bought exactly half as much fruit as Bob. But she paid 2.77 when half of Bob's payment would have been only 2.63. The third row of `A` is equal to one-half of the second row, but b(3) is not equal to one-half of b(2). For this particular matrix A and vector b, the solution to the linear system of equations $Ax = b$ does not exist.

What if we make Carol's purchase price consistent with Bob's? We leave `A` unchanged and revise `b` with

```
b(3) = 2.63
```

so

```
b =
    2.36
    5.26
    2.63
```

Now we do not have enough information. Our last two equations are scalar multiples of each other.

$$\begin{aligned} 12x_1 + 2x_3 &= 5.26 \\ 6x_1 + \ x_3 &= 2.63 \end{aligned}$$

We can pick an arbitrary value for x_1 or x_3, use either of these equations to compute the value we didn't pick, and then use the first equation to compute x_2. The result is a solution to all three equations. One possible solution is the solution to the original problem.

```
x =
    0.29
    0.05
    0.89
```

But another solution is

```
x =
    0.09
       0
    2.09
```

There are infinitely many more. In this case, for this particular matrix A and vector b, the solution to $Ax = b$ is not unique.

This illustrates two fundamental facts about technical computing.

- The hardest things to compute are things that do not exist.
- The next hardest things to compute are things that are not unique.

Exercises

11.1 *Two-by-two.* Use backslash to try to solve each of these systems of equations. Indicate if the solution exists, and if it unique.

(a)

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} x = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

(d)

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

11.2 *Three-by-three.* Use backslash to try to solve each of these systems of equations. Indicate if the solution exists, and if it unique.

(a)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 12 & 6 \\ 7 & 8 & 12 \end{pmatrix} x = \begin{pmatrix} 3 \\ 12 \\ 15 \end{pmatrix}$$

(d)

$$\begin{pmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(e)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(f)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

11.3 *Null vector.* Find a nonzero solution x to

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

11.4 Matrix equations. Backslash can be used to solve matrix equations of the form

$$AX = B$$

where B has several colums. Do you recognize the solution to the following equation?

$$\begin{pmatrix} 53 & -52 & 23 \\ -22 & 8 & 38 \\ -7 & 68 & -37 \end{pmatrix} X = \begin{pmatrix} 360 & 0 & 0 \\ 0 & 360 & 0 \\ 0 & 0 & 360 \end{pmatrix}$$